Solitary Wave Families in Two Non-Integrable Models Using Reversible Systems Theory

Jonathan Leto

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Overview

• Definitions
• Background
• The Generalized Pochhammer-Chree Equations
• A Generalized Microstructure Equation
Reversible Dynamical System
(Iooss & Adelmeyer)

Consider

\[
\frac{dz}{dt} = F(z; \mu), \ z \in \mathbb{R}^n, \mu \in \mathbb{R}
\]  \hspace{1cm} (1)

where

\[ F(0; 0) = 0 \]

. If there exists a **unitary map**

\[ S : \mathbb{R}^n \rightarrow \mathbb{R}^n, \ S \neq I \]

such that

\[ F(Sz; \mu) = -SF(z; \mu) \]

for all \( z \) and \( \mu \) then (1) is a reversible system.
Here we will use the term solitary wave or "soliton" to mean a pulse-like solution to an evolution equation. For example,

$$A(z) = \ell \text{sech}^2 kz$$

is a two-parameter family of solitary waves. where $k$ and $\ell$ are parameters which determine the speed and the height of the wave.
Normal Form Theory

After a nonlinear change of variables (Iooss & Adelmeyer) one may put the Center Manifold into Normal Form.

**Two-Dimensional Center Manifold**
- \( \lambda_1 = 0, 0, \pm \lambda, \lambda \in \mathbb{R} \)
- \( Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B) \)

**Four-Dimensional Center Manifold**
- \( \lambda_1 = 0, 0, \pm i\omega, \omega \in \mathbb{R} \)
- \( Y = A\zeta_0 + B\zeta_1 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C}) \)

where \( \zeta_0, \zeta_1, \zeta_+, \zeta_- \) are eigenvectors of the linearized operator.
Properties of Bilinear Functions

A function

\[ B : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \]

satisfying the following axioms

\[
\begin{align*}
B(x + y, z) &= B(x, z) + B(y, z) \\
B(\lambda x, y) &= \lambda B(x, y) \\
B(x, y + z) &= B(x, y) + B(x, z) \\
B(x, \lambda y) &= \lambda B(x, y)
\end{align*}
\]

is called \textit{bilinear}. If \( B(x, y) \) is bilinear, then \( f(y) \equiv B(y, y) \) is invariant under the transformation \( y \mapsto -y \) and thus is \textit{reversible}. 
The Generalized Pochhammer-Chree Equations

The Generalized Pochhammer-Chree Equations govern the propagation of longitudinal waves in elastic rods.

- **GPC1**
  \[(u - u_{xx})_{tt} - (a_1 u + a_2 u^2 + a_3 u^3)_{xx} = 0\]  \(\text{(4)}\)
- **GPC2**
  \[(u - u_{xx})_{tt} - (a_1 u + a_3 u^3 + a_5 u^5)_{xx} = 0\]  \(\text{(5)}\)
Let $z = x - ct$ and $u(x, t) = \phi(z)$ to reduce (4) and (5) to the Travelling Wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}_{1,2}[\phi]$$

where

$$p \equiv 0$$

$$q \equiv 1 - \frac{a_1}{c^2}$$

$$L_{pq} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
q/3 & 0 & 1 & 0 \\
0 & q/3 & 0 & 1 \\
q^2 - p & 0 & q/3 & 0
\end{pmatrix}$$

$$\mathcal{N}_1[\phi] = -\frac{1}{c^2} \left[3a_3 \left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 2a_2 \left(\phi_{zz}\phi_z + \phi_z^2\right)\right]$$

$$\mathcal{N}_2[\phi] = -\frac{1}{c^2} \left[3a_3 \left(2\phi\phi_z^2 + \phi^2\phi_{zz}\right) + 5a_5 \left(4\phi^3\phi_z^2 + \phi^4\phi_{zz}\right)\right]$$
Denoting $Y = \langle y_1, y_2, y_3, y_4 \rangle^T = \langle \phi, \phi_z, \phi_{zz}, \phi_{zzz} \rangle^T$ equation (6) can be written

$$\frac{dY}{dz} = L_{pq} Y - G_{1,2}(Y, Y)$$  \hspace{1cm} (9)$$

where

$$L_{pq} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
q/3 & 0 & 1 & 0 \\
0 & q/3 & 0 & 1 \\
q^2 - p & 0 & q/3 & 0
\end{pmatrix}$$

$$G_{1,2}(Y, Y) = \langle 0, 0, 0, -N_{1,2}(Y, Y) \rangle^T$$
Near $C_0$: Normal Form

The eigenvalues are $\lambda_{1,4} = 0, 0, \pm \lambda, \lambda \in \mathbb{R}$, We write the Center Manifold

$$\mathcal{Y} = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$$  \hspace{1cm} (10)

with corresponding normal form

$$\frac{dA}{dz} = B$$ \hspace{1cm} (11a)
$$\frac{dB}{dz} = b\epsilon A + \tilde{c}A^2$$ \hspace{1cm} (11b)

where $\epsilon$ measures the perturbation about $C_0$ and

$$\zeta_0 = \langle 1, 0, -q/3, 0 \rangle^T$$ \hspace{1cm} (12a)
$$\zeta_1 = \langle 0, 1, 0, -2q/3 \rangle^T$$ \hspace{1cm} (12b)

How do we determine the coefficients $b$ and $\tilde{c}$?
Finding Coefficients Of The Normal Form

When you follow two separate chains of thought, Watson, you will find some point of intersection which should approximate the truth. - Sir Arthur Conan Doyle

By computing $\frac{dY}{dz}$ in two ways and comparing the coefficients of each term in the normal form, we find systems of equations which determine each coefficient. For example, to find $\tilde{c}$, we compare $O(A^2)$ terms.
Two Ways to Compute $\frac{dY}{dz}$

**Method 1**
Use the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$$

in the reversible system

$$\frac{dY}{dz} = L_{pq} Y - G_{1,2}(Y, Y)$$

and repeatedly use the bilinear properties of $G_{1,2}$ to simplify.
Two Ways to Compute $\frac{dY}{dz}$

**Method 1**

Use the Center Manifold

\[ Y = A\zeta_0 + B\zeta_1 + \psi(\epsilon, A, B) \]

in the reversible system

\[ \frac{dY}{dz} = L_{pq}Y - G_{1,2}(Y, Y) \]

and repeatedly use the bilinear properties of $G_{1,2}$ to simplify.

where

\[ \psi(\epsilon, A, B) = \epsilon A\psi_{10}^1 + \epsilon B\psi_{01}^1 + A^2\psi_{20}^0 + AB\psi_{11}^0 + B^2\psi_{02}^0 + \cdots \]

**Method 2**

Differentiate the Center Manifold

\[ Y = A\zeta_0 + B\zeta_1 + \psi(\epsilon, A, B) \]

and use the Normal Form

\[ \frac{dA}{dz} = B \]

\[ \frac{dB}{dz} = b\epsilon A + \tilde{c}A^2 \]

to simplify all derivatives.
Matching $O(A^2)$ terms in each method gives us the two systems of equations

**GPC 1**

\[ \tilde{c} \zeta_1 = L_0 \psi_{20}^0 - G_1 (\zeta_0, \zeta_0) \]

\[
\begin{align*}
0 &= \frac{q}{3} x_1 + x_3 \\
\tilde{c} &= \frac{q}{3} x_2 + x_4 \implies x_4 = 0 \\
- \frac{2q}{3} \tilde{c} &= \frac{q}{3} \left( \frac{q}{3} x_1 + x_3 \right) + \frac{q}{3 c^2} (3a_3 + 5a_5) \\
&= \frac{q}{3} \tilde{c} + \frac{q}{3 c^2} a_3 \\
&\implies \tilde{c} = - \frac{a_3}{3 c^2}
\end{align*}
\]
Finding $\tilde{c}$

Matching $\mathcal{O}(A^2)$ terms in each method gives us the two systems of equations

**GPC 1**

$$\ddot{\zeta}_1 = L_0 q \psi_{20}^0 - G_1 (\zeta_0, \zeta_0)$$

$$0 = x_2$$

$$\tilde{c} = \frac{q}{3} x_1 + x_3$$

$$0 = \frac{q}{3} x_2 + x_4 \implies x_4 = 0$$

$$-2q \tilde{c} = \frac{q}{3} \left( \frac{q}{3} x_1 + x_3 \right) + \frac{q}{3c^2} (3a_3 + 5a_5)$$

$$= \frac{q}{3} \tilde{c} + \frac{q}{3c^2} a_3$$

$$\implies \tilde{c} = -\frac{a_3}{3c^2}$$

**GPC 2**

$$\ddot{\zeta}_1 = L_0 q \psi_{20}^0 - G_2 (\zeta_0, \zeta_0)$$

$$0 = x_2$$

$$\tilde{c} = \frac{q}{3} x_1 + x_3$$

$$0 = \frac{q}{3} x_2 + x_4 \implies x_4 = 0$$

$$-2q \tilde{c} = \frac{q}{3} \left( \frac{q}{3} x_1 + x_3 \right) + \frac{q}{3c^2} (3a_3 + 5a_5)$$

$$= \frac{q}{3} \tilde{c} + \frac{q}{3c^2} (3a_3 + 5a_5)$$

$$\implies \tilde{c} = -\frac{1}{3c^2} (3a_3 + 5a_5)$$

where $\psi_{20}^0 = (x_1, x_2, x_3, x_4)^T$. 
Therefore the Normal Form near \( C_0 \) is

**GPC 1**

\[
\begin{align*}
\frac{dA}{dz} & = B \\
\frac{dB}{dz} & = -\frac{\epsilon}{q} A - \frac{a_3}{c^2} A^2
\end{align*}
\]

**GPC 2**

\[
\begin{align*}
\frac{dA}{dz} & = B \\
\frac{dB}{dz} & = -\frac{\epsilon}{q} A - \frac{1}{3c^2} (3a_3 + 5a_5) A^2
\end{align*}
\]

These equations admit homoclinic solutions near \( C_0 \) of the form

\[A(z) = \ell \text{sech}^2 (kz)\]
Finding $k$ and $\ell$

To determine $k$ and $\ell$, we first write the Normal Form as a single second order equation. Then we use our expression for $A(z)$ and compare coefficients of $O(\sech^2(kz))$ and $O(\sech^4(kz))$ which implies

\[
\text{GPC 1} \quad k = \sqrt{\frac{-\epsilon}{4q}} \\
\ell = \frac{-3\epsilon c^2}{2qa_3}
\]

\[
\text{GPC 2} \quad k = \sqrt{\frac{-\epsilon}{4q}} \\
\ell = \frac{-9\epsilon c^2}{2q(3a_3 + 5a_5)}
\]

Hence, since $\epsilon = -p$, solitons of this form exist for $p = 0^+, q > 0$, which implies $a_1 < c^2$. 
Near $C_1$: Normal Form

The eigenvalues are $\lambda_{1,4} = 0, 0, \pm i\omega, \omega \in \mathbb{R}$, We write the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C})$$

(20)

with corresponding normal form

$$\begin{align*}
\frac{dA}{dz} &= B \\
\frac{dB}{dz} &= \bar{\nu}\epsilon A + b_*A^2 + c_*\|C\|^2 \\
\frac{dC}{dz} &= i\nu_0 C + i\bar{\nu}d_1 C + i\nu d_2 AC
\end{align*}$$

where the new eigenvectors co-spanning the four-dimensional Center Manifold are

$$\zeta_{\pm} = \langle 1, \lambda_{\pm}, 2q/3, \lambda_{\pm}q/3 \rangle^T$$

and $\lambda_{\pm} = \pm i\sqrt{-q}, q < 0.$
Compute \( \frac{dY}{dz} \) by both methods

Matching the coefficients of \( A^2, \epsilon C \| C \|^2 \) and \( AC \) in the two separate expressions for \( dY/dz \) yields the following two systems of equations:

\[
\mathcal{O}(A^2) : \quad b_\ast \zeta_1 = L_0q \psi_{2000}^0 - G_{1,2}(\zeta_0, \zeta_0) \\
\mathcal{O}(\| C \|^2) : \quad c_\ast \zeta_1 = L_0q \psi_{0011}^0 - 2G_{1,2}(\zeta_+, \zeta_-) \\
\mathcal{O}(\epsilon C) : \quad -\frac{i}{q} \left( d_1 \zeta_+ + d_0 \psi_{0010}^1 \right) = L_0q \psi_{0010}^1 \\
\mathcal{O}(AC) : \quad id_2 \zeta_+ + id_0 \psi_{1101}^{00} = L_0q \psi_{1010}^0 - 2G_{1,2}(\zeta_0, \zeta_+) \]
Normal Form near $C_1 : GPC 1$

The resolution of the preceding equations yields the coefficients in the Normal Form for the GPC 1 equation:

\[
\begin{align*}
\frac{dA}{dz} &= B \\
\frac{dB}{dz} &= -\frac{\epsilon}{q} A + \frac{a_3}{c^2} A^2 + \frac{1}{c^2} \left( \frac{7}{2\sqrt{-q}} a_3 - \frac{i}{3} a_2 \right) |C|^2 \\
\frac{dC}{dz} &= i\sqrt{-q} C - i\frac{\sqrt{-q}}{q^3} C\epsilon + i\frac{1}{c^2} \left( \frac{7}{2\sqrt{-q}} a_3 - \frac{i}{3} a_2 \right) AC
\end{align*}
\]

where $q \equiv 1 - \frac{a_1}{c^2}$
Normal Form near $C_1$: GPC 2

The resolution of the preceding equations yields the coefficients in the Normal Form for the GPC 2 equation:

\[
\begin{align*}
\frac{dA}{dz} &= B \\
\frac{dB}{dz} &= -\frac{\epsilon}{q} A + \frac{1}{3c^2} (3a_3 + 5a_5) A^2 + \frac{1}{c^2} \left(16a_3 + \frac{140}{3} a_5\right) |C|^2 \\
\frac{dC}{dz} &= i\sqrt{-q} C - i\frac{\sqrt{-q}}{q^3} C \epsilon + i \frac{1}{\sqrt{-qc^2}} \left(\frac{7}{2} a_3 + \frac{32}{3} a_5\right) AC
\end{align*}
\]

where $q \equiv 1 - \frac{a_1}{c^2}$
Dynamics Near $C_1$

The two first integrals of the four-dimensional Normal Form are

$$K = \|C\|^2$$

and

$$H = B^2 - \frac{2}{3}b_*A^2 - \bar{v}A^2 - 2c_*KA$$

In the $(A, B)$ phase plane, the level curve $H = 0$ compromises a homoclinic orbit. The intersection of $H = 0$ with the $A$ axis occurs for

$$A_{\mp} = \frac{3}{4b_*} \left[ \bar{v} \pm \sqrt{\bar{v}^2 + \frac{16b_*c_*K}{3}} \right]$$
Homoclinic Orbits for Various Values of H

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Solitary Wave
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GPC
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\[ b_* = -0.11 \quad \nu = -0.994 \quad c_* = -1.05 \quad K = 0.01 \]

\[ b_* = -0.11 \quad \nu = -0.975 \quad c_* = -1.05 \quad K = 0.01 \]

\[ b_* = -0.11 \quad \nu = -0.961 \quad c_* = -1.05 \quad K = 0.01 \]

\[ b_* = -0.11 \quad \nu = -0.952 \quad c_* = -1.05 \quad K = 0.01 \]
A Generalized Microstructure PDE

One dimensional wave propagation in microstructured solids has recently been modeled by an equation

\[ v_{tt} - bv_{xx} - \frac{\mu}{2} (v^2)_{xx} - \delta (\beta v_{tt} - \gamma v_{xx})_{xx} = 0 \]  (25)
Let \( z = x - ct \) and \( u(x, t) = \phi(z) \) to reduce (25) to the Travelling Wave ODE

\[
\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}[\phi]
\]

where

\[
\mathcal{N}[\phi] = -\Delta_1 \phi_z^2 - b\Delta_1 \phi \phi_{zz}
\]

\[
\begin{align*}
z &\equiv x - ct \\
p &\equiv 0 \\
q &\equiv \frac{c^2 - b}{\delta (\beta c^2 - \gamma)} \\
\Delta_1 &\equiv \frac{\mu}{\delta (\beta c^2 - \gamma)}
\end{align*}
\]
Normal Form near $C_0$

With the same kind analysis we find

$$\frac{dA}{dz} = B$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2$$
The Normal Form admits a homoclinic solution near $C_0$ of the form

$$A(z) = \ell \text{sech}^2(kz)$$

where

$$k = \sqrt{\frac{-\epsilon}{4q}}$$

$$\ell = \frac{6k^2}{b\Delta_1}$$

Since $\epsilon = -p$, and the curve $C_0$ corresponds to $p = 0, q > 0$, solitary waves exist near $C_0$ for $p > 0, q > 0$, which implies that $\frac{c^2-b}{\delta(\beta c^2-\gamma)} > 0$. 

Solitary Waves Near $C_0$
Near $C_1$ the Normal Form for (25) is

\[
\begin{align*}
\frac{dA}{dz} &= B \\
\frac{dB}{dz} &= -\frac{\epsilon}{q} A - \frac{b\Delta_1}{3} A^2 + 2\Delta_1 \left( \frac{2b}{3} - 1 \right) |C|^2 \\
\frac{dC}{dz} &= i\sqrt{-q} C - i\frac{\sqrt{-q}}{q^3} C \epsilon + i \frac{b\Delta_1}{6\sqrt{-q}} AC
\end{align*}
\]
Homoclinic Orbits for Various Values of H
Open Problems

Study embedded solitons using a mix of

- exponential asymptotics
- numerical shooting
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