

J.A. Leto

Overview

Definitions

Reversible  
System

Solitary Wave

Background

Normal Forms

Bilinear  
Functions

GPC

MS

# Solitary Wave Families in Two Non-Integrable Models Using Reversible Systems Theory

Jonathan Leto

April 3, 2008

# Overview

- Definitions
- Background
- The Generalized Pochhammer-Chree Equations
- A Generalized Microstructure Equation

# Reversible Dynamical System (Iooss & Adelmeyer)

Consider

$$\frac{dz}{dt} = F(z; \mu), z \in \mathbb{R}^n, \mu \in \mathbb{R} \quad (1)$$

where

$$F(0; 0) = 0$$

. If there exists a **unitary map**

$$S : \mathbb{R}^n \mapsto \mathbb{R}^n, S \neq I$$

such that

$$F(Sz; \mu) = -SF(z; \mu)$$

for all  $z$  and  $\mu$  then (1) is a reversible system.

# Families of Solitary Waves

Here we will use the term solitary wave or "soliton" to mean a pulse-like solution to an evolution equation. For example,

$$A(z) = \ell \operatorname{sech}^2 kz \quad (2)$$

is a two-parameter family of solitary waves.

where  $k$  and  $\ell$  are parameters which determine the speed and the height of the wave.

# Normal Form Theory

After a nonlinear change of variables (Iooss & Adelmeyer) one may put the Center Manifold into Normal Form.

## Two-Dimensional Center Manifold

- $\lambda_{1-4} = 0, 0, \pm\lambda, \lambda \in \mathbb{R}$
- $Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$

## Four-Dimensional Center Manifold

- $\lambda_{1-4} = 0, 0, \pm i\omega, \omega \in \mathbb{R}$
- $Y = A\zeta_0 + B\zeta_1 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C})$

where  $\zeta_0, \zeta_1, \zeta_+, \zeta_-$  are eigenvectors of the linearized operator.

# Properties of Bilinear Functions

A function

$$B : \mathbb{C} \times \mathbb{C} \mapsto \mathbb{C}$$

satisfying the following axioms

$$B(x + y, z) = B(x, z) + B(y, z)$$

$$B(\lambda x, y) = \lambda B(x, y)$$

$$B(x, y + z) = B(x, y) + B(x, z)$$

$$B(x, \lambda y) = \lambda B(x, y)$$

is called **bilinear** .

If  $B(x, y)$  is bilinear, then  $f(y) \equiv B(y, y)$  is invariant under the transformation  $y \mapsto -y$  and thus is **reversible** .

# The Generalized Pochhammer-Chree Equations

The Generalized Pochhammer-Chree Equations govern the propagation of longitudinal waves in elastic rods.

- GPC1

$$(u - u_{xx})_{tt} - (a_1 u + a_2 u^2 + a_3 u^3)_{xx} = 0 \quad (4)$$

- GPC2

$$(u - u_{xx})_{tt} - (a_1 u + a_3 u^3 + a_5 u^5)_{xx} = 0 \quad (5)$$

## Travelling Wave ODE

Let  $z = x - ct$  and  $u(x, t) = \phi(z)$  to reduce (4) and (5) to the Travelling Wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}_{1,2}[\phi] \quad (6)$$

where

$$p \equiv 0$$

$$q \equiv 1 - \frac{a_1}{c^2}$$

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix} \quad (8a)$$

$$\mathcal{N}_1[\phi] = -\frac{1}{c^2} [3a_3 (2\phi\phi_z^2 + \phi^2\phi_{zz}) + 2a_2 (\phi_{zz}\phi_z + \phi_z^2)]$$

$$\mathcal{N}_2[\phi] = -\frac{1}{c^2} [3a_3 (2\phi\phi_z^2 + \phi^2\phi_{zz}) + 5a_5 (4\phi^3\phi_z^2 + \phi^4\phi_{zz})]$$

## Reversible Form

Denoting  $Y = \langle y_1, y_2, y_3, y_4 \rangle^T = \langle \phi, \phi_z, \phi_{zz}, \phi_{zzz} \rangle^T$  equation (6) can be written

$$\frac{dY}{dz} = L_{pq} Y - G_{1,2}(Y, Y) \quad (9)$$

where

$$L_{pq} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q/3 & 0 & 1 & 0 \\ 0 & q/3 & 0 & 1 \\ q^2 - p & 0 & q/3 & 0 \end{pmatrix}$$

$$G_{1,2}(Y, Y) = \langle 0, 0, 0, -\mathcal{N}_{1,2}(Y, Y) \rangle^T$$

## Near $C_0$ : Normal Form

The eigenvalues are  $\lambda_{1,4} = 0, 0, \pm\lambda$ ,  $\lambda \in \mathbb{R}$ , We write the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B) \quad (10)$$

with corresponding normal form

$$\frac{dA}{dz} = B \quad (11a)$$

$$\frac{dB}{dz} = b\epsilon A + \tilde{c}A^2 \quad (11b)$$

where  $\epsilon$  measures the perturbation about  $C_0$  and

$$\zeta_0 = \langle 1, 0, -q/3, 0 \rangle^T \quad (12a)$$

$$\zeta_1 = \langle 0, 1, 0, -2q/3 \rangle^T \quad (12b)$$

How do we determine the coefficients  $b$  and  $\tilde{c}$  ?

# Finding Coefficients Of The Normal Form

*When you follow two separate chains of thought, Watson, you will find some point of intersection which should approximate the truth. - Sir Arthur Conan Doyle*

By computing  $\frac{dY}{dz}$  in two ways and comparing the coefficients of each term in the normal form, we find systems of equations which determine each coefficient. For example, to find  $\tilde{c}$ , we compare  $\mathcal{O}(A^2)$  terms.

# Two Ways to Compute $\frac{dY}{dz}$

## Method 1

Use the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$$

in the reversible system

$$\frac{dY}{dz} = L_{pq}Y - G_{1,2}(Y, Y)$$

and repeatedly use the bilinear  
properties of  $G_{1,2}$  to simplify.

# Two Ways to Compute $\frac{dY}{dz}$

## Method 1

Use the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$$

in the reversible system

$$\frac{dY}{dz} = L_{pq} Y - G_{1,2}(Y, Y)$$

and repeatedly use the bilinear  
properties of  $G_{1,2}$  to simplify.

where

$$\Psi(\epsilon, A, B) = \epsilon A \Psi_{10}^1 + \epsilon B \Psi_{01}^1 + A^2 \Psi_{20}^0 + AB \Psi_{11}^0 + B^2 \Psi_{02}^0 + \dots$$

## Method 2

Differentiate the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + \Psi(\epsilon, A, B)$$

and use the Normal Form

$$\frac{dA}{dz} = B$$

$$\frac{dB}{dz} = b\epsilon A + \tilde{c}A^2$$

to simplify all derivatives.

## Finding $\tilde{c}$

Matching  $\mathcal{O}(A^2)$  terms in each method gives us the two systems of equations

### GPC 1

$$\tilde{c}\zeta_1 = L_{0q}\psi_{20}^0 - G_1(\zeta_0, \zeta_0)$$

$$0 = x_2$$

$$\tilde{c} = \frac{q}{3}x_1 + x_3$$

$$0 = \frac{q}{3}x_2 + x_4 \implies x_4 = 0$$

$$-\frac{2q}{3}\tilde{c} = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{q}{3c^2}(3a_3 + 5a_5)$$

$$= \frac{q}{3}\tilde{c} + \frac{q}{3c^2}a_3$$

$$\implies \tilde{c} = -\frac{a_3}{3c^2}$$

## Finding $\tilde{c}$

Matching  $\mathcal{O}(A^2)$  terms in each method gives us the two systems of equations

### GPC 1

$$\tilde{c}\zeta_1 = L_{0q}\psi_{20}^0 - G_1(\zeta_0, \zeta_0)$$

$$0 = x_2$$

$$\tilde{c} = \frac{q}{3}x_1 + x_3$$

$$0 = \frac{q}{3}x_2 + x_4 \implies x_4 = 0$$

$$-\frac{2q}{3}\tilde{c} = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{q}{3c^2}(3a_3 + 5a_5)$$

$$= \frac{q}{3}\tilde{c} + \frac{q}{3c^2}a_3$$

$$\implies \tilde{c} = -\frac{a_3}{3c^2}$$

### GPC 2

$$\tilde{c}\zeta_1 = L_{0q}\psi_{20}^0 - G_2(\zeta_0, \zeta_0)$$

$$0 = x_2$$

$$\tilde{c} = \frac{q}{3}x_1 + x_3$$

$$0 = \frac{q}{3}x_2 + x_4 \implies x_4 = 0$$

$$-\frac{2q}{3}\tilde{c} = \frac{q}{3}\left(\frac{q}{3}x_1 + x_3\right) + \frac{q}{3c^2}(3a_3 + 5a_5)$$

$$= \frac{q}{3}\tilde{c} + \frac{q}{3c^2}(3a_3 + 5a_5)$$

$$\implies \tilde{c} = -\frac{1}{3c^2}(3a_3 + 5a_5)$$

where  $\Psi_{20}^0 = \langle x_1, x_2, x_3, x_4 \rangle^T$ .

## Normal Form Near $C_0$

Therefore the Normal Form near  $C_0$  is

**GPC 1**

$$\begin{aligned}\frac{dA}{dz} &= B \\ \frac{dB}{dz} &= -\frac{\epsilon}{q}A - \frac{a_3}{c^2}A^2\end{aligned}$$

**GPC 2**

$$\begin{aligned}\frac{dA}{dz} &= B \\ \frac{dB}{dz} &= -\frac{\epsilon}{q}A - \frac{1}{3c^2}(3a_3 + 5a_5)A^2\end{aligned}$$

These equations admit homoclinic solutions near  $C_0$  of the form

$$A(z) = l \operatorname{sech}^2(kz)$$

## Finding $k$ and $\ell$

To determine  $k$  and  $\ell$ , we first write the Normal Form as a single second order equation. Then we use our expression for  $A(z)$  and compare coefficients of  $\mathcal{O}(\operatorname{sech}^2(kz))$  and  $\mathcal{O}(\operatorname{sech}^4(kz))$  which implies

### GPC 1

$$k = \sqrt{\frac{-\epsilon}{4q}}$$

$$\ell = \frac{-3\epsilon c^2}{2qa_3}$$

### GPC 2

$$k = \sqrt{\frac{-\epsilon}{4q}}$$

$$\ell = \frac{-9\epsilon c^2}{2q(3a_3 + 5a_5)}$$

Hence, since  $\epsilon = -p$ , solitons of this form exist for  $p = 0^+$ ,  $q > 0$ , which implies  $a_1 < c^2$ .

## Near $C_1$ : Normal Form

The eigenvalues are  $\lambda_{1,4} = 0, 0, \pm i\omega$ ,  $\omega \in \mathbb{R}$ , We write the Center Manifold

$$Y = A\zeta_0 + B\zeta_1 + C\zeta_+ + \bar{C}\zeta_- + \Psi(\epsilon, A, B, C, \bar{C}) \quad (20)$$

with corresponding normal form

$$\begin{aligned} \frac{dA}{dz} &= B \\ \frac{dB}{dz} &= \bar{\nu}\epsilon A + b_* A^2 + c_* \|C\|^2 \\ \frac{dC}{dz} &= id_0 C + i\bar{\nu}d_1 C + id_2 AC \end{aligned}$$

where the new eigenvectors co-spanning the four-dimensional Center Manifold are

$$\zeta_{\pm} = \langle 1, \lambda_{\pm}, 2q/3, \lambda_{\pm}q/3 \rangle^T$$

and  $\lambda_{\pm} = \pm i\sqrt{-q}$ ,  $q < 0$ .

## Compute $\frac{dY}{dz}$ by both methods

Matching the coefficients of  $A^2$ ,  $\epsilon C \|C\|^2$  and  $AC$  in the two separate expressions for  $dY/dz$  yields the following two systems of equations:

$$\begin{aligned}
 \mathcal{O}(A^2) : \quad & b_* \zeta_1 & = L_{0q} \Psi_{2000}^0 - G_{1,2}(\zeta_0, \zeta_0) \\
 \mathcal{O}(|C|^2) : \quad & c_* \zeta_1 & = L_{0q} \Psi_{0011}^0 - 2G_{1,2}(\zeta_+, \zeta_-) \\
 \mathcal{O}(\epsilon C) : \quad & -\frac{i}{q} (d_1 \zeta_+ + d_0 \Psi_{0010}^1) & = L_{0q} \Psi_{0010}^1 \\
 \mathcal{O}(AC) : \quad & id_2 \zeta_+ + id_0 \Psi_{1010}^0 & = L_{0q} \Psi_{1010}^0 - 2G_{1,2}(\zeta_0, \zeta_+)
 \end{aligned}$$

## Normal Form near $C_1$ : GPC 1

The resolution of the preceding equations yields the coefficients in the Normal Form for the GPC 1 equation:

$$\frac{dA}{dz} = B$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A + \frac{a_3}{c^2}A^2 + \frac{1}{c^2} \left( \frac{7}{2\sqrt{-q}}a_3 - \frac{i}{3}a_2 \right) |C|^2$$

$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{1}{c^2} \left( \frac{7}{2\sqrt{-q}}a_3 - \frac{i}{3}a_2 \right) AC$$

where  $q \equiv 1 - \frac{a_1}{c^2}$

## Normal Form near $C_1$ : GPC 2

The resolution of the preceding equations yields the coefficients in the Normal Form for the GPC 2 equation:

$$\frac{dA}{dz} = B$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A + \frac{1}{3c^2}(3a_3 + 5a_5)A^2 + \frac{1}{c^2}\left(16a_3 + \frac{140}{3}a_5\right)|C|^2$$

$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{1}{\sqrt{-q}c^2}\left(\frac{7}{2}a_3 + \frac{32}{3}a_5\right)AC$$

where  $q \equiv 1 - \frac{a_1}{c^2}$

## Dynamics Near $C_1$

The two first integrals of the four-dimensional Normal Form are

$$K = \|C\|^2$$

and

$$H = B^2 - \frac{2}{3}b_*A^2 - \bar{\nu}A^2 - 2c_*KA$$

In the  $(A, B)$  phase plane, the level curve  $H = 0$  comprises a

homoclinic orbit. The intersection of  $H = 0$  with the  $A$  axis occurs for

$$A_{\mp} = \frac{3}{4b_*} \left[ \bar{\nu} \pm \sqrt{\bar{\nu}^2 + \frac{16b_*c_*K}{3}} \right]$$

J.A. Leto

# Homoclinic Orbits for Various Values of H

Overview

Definitions

Reversible  
System

Solitary Wave

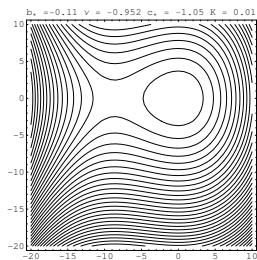
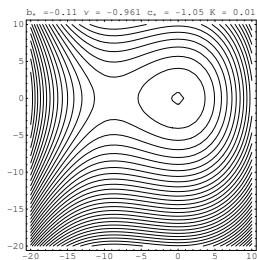
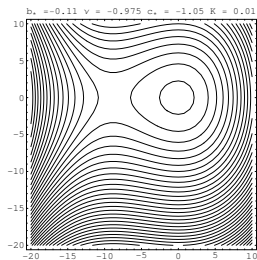
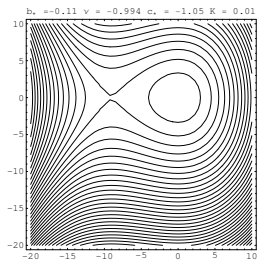
Background

Normal Forms

Bilinear  
Functions

GPC

MS



# A Generalized Microstructure PDE

J.A. Leto

Overview

Definitions

Reversible  
System

Solitary Wave

Background

Normal Forms

Bilinear  
Functions

GPC

MS

One dimensional wave propagation in microstructured solids has recently been modeled by an equation

$$v_{tt} - bv_{xx} - \frac{\mu}{2} (v^2)_{xx} - \delta (\beta v_{tt} - \gamma v_{xx})_{xx} = 0 \quad (25)$$

## Travelling Wave ODE

Let  $z = x - ct$  and  $u(x, t) = \phi(z)$  to reduce (25) to the Travelling Wave ODE

$$\phi_{zzzz} - q\phi_{zz} + p\phi = \mathcal{N}[\phi] \quad (26)$$

where

$$\mathcal{N}[\phi] = -\Delta_1\phi_z^2 - b\Delta_1\phi\phi_{zz} \quad (27)$$

$$z \equiv x - ct$$

$$p \equiv 0$$

$$q \equiv \frac{c^2 - b}{\delta(\beta c^2 - \gamma)}$$

$$\Delta_1 \equiv \frac{\mu}{\delta(\beta c^2 - \gamma)}$$

## Normal Form near $C_0$

With the same kind analysis we find

$$\begin{aligned}\frac{dA}{dz} &= B \\ \frac{dB}{dz} &= -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2\end{aligned}$$

## Solitary Waves Near $C_0$

The Normal Form admits a homoclinic solution near  $C_0$  of the form

$$A(z) = \ell \operatorname{sech}^2(kz)$$

where

$$k = \sqrt{\frac{-\epsilon}{4q}}$$
$$\ell = \frac{6k^2}{b\Delta_1}$$

Since  $\epsilon = -p$ , and the curve  $C_0$  corresponds to  $p = 0, q > 0$ , solitary waves exist near  $C_0$  for  $p > 0, q > 0$ , which implies that  $\frac{c^2 - b}{\delta(\beta c^2 - \gamma)} > 0$ .

## Normal Form near $C_1$

Near  $C_1$  the Normal Form for (25) is

$$\frac{dA}{dz} = B$$

$$\frac{dB}{dz} = -\frac{\epsilon}{q}A - \frac{b\Delta_1}{3}A^2 + 2\Delta_1 \left( \frac{2b}{3} - 1 \right) |C|^2$$

$$\frac{dC}{dz} = i\sqrt{-q}C - i\frac{\sqrt{-q}}{q^3}C\epsilon + i\frac{b\Delta_1}{6\sqrt{-q}}AC$$

J.A. Leto

# Homoclinic Orbits for Various Values of H

Overview

Definitions

Reversible  
System

Solitary Wave

Background

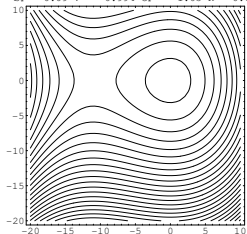
Normal Forms

Bilinear  
Functions

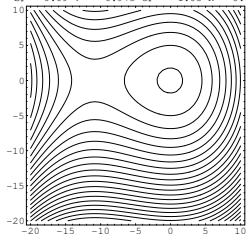
GPC

MS

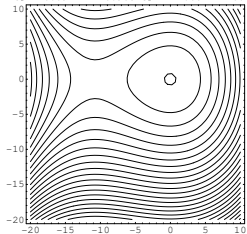
$b_s = -0.09$   $v = -0.994$   $c_s = -1.05$   $K = 0.01$



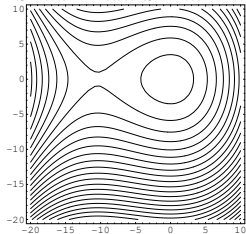
$b_s = -0.09$   $v = -0.975$   $c_s = -1.05$   $K = 0.01$



$b_s = -0.09$   $v = -0.968$   $c_s = -1.05$   $K = 0.01$



$b_s = -0.09$   $v = -0.938$   $c_s = -1.05$   $K = 0.01$



# Open Problems

Study embedded solitons using a mix of

- exponential asymptotics
- numerical shooting

# Thanks

- Dr. Choudhury, Dr. Mohapatra, Dr. Rollins for being great committee members
- Erin Langsdorf for being my Orlando graduation liaison